Sudarshan-Glauber *P*, *Q*, and Wigner *W* Distributions in Correlated-Spontaneous-Emission Lasers *

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A comparative study of Sudarshan-Glauber *P*, *Q* and Wigner *W* distributions in correlated-spontaneous-emission lasers has been made. Immediately giving the information of squeezing by examining the diffusion coefficients is the advantage of the Sudarshan-Glauber *P* representation.

Summary

Recently, several mechanisms for the correlated-emission laser (CEL) have been considered [1–4]. The correlated emission is based on using atoms prepared in a coherent superposition of the states between which the laser emission takes place. The initial atomic coherence can lead to the reduction in either place or amplitude noise. It can even lead to the squeezing in one of the quadratures of the field. The microscopic theories of the (single-mode) CEL show that the master equation for the field mode a can be written in the form [1]

$$\begin{aligned}\dot{\varrho} &= \Lambda_1(\alpha^\dagger \varrho a - \varrho a a^\dagger) + \Lambda_2(a p a^\dagger - a^\dagger a \varrho) \\ &\quad + \Lambda_3(\varrho a^{\dagger 2} - a^\dagger \varrho a^\dagger) + \Lambda_4(a^{\dagger 2}\varrho - a^\dagger \varrho a^\dagger) \\ &\quad + f\,[a^\dagger, \varrho] + H.c.\,, \end{aligned} \tag{1}$$

where $f = -is(\varrho_{ab} + \varrho_{bc})$ and

$$\begin{aligned}\Lambda_1 &= \tfrac{1}{2}\alpha_0(\varrho_{aa} + \varrho_{bb} - \tfrac{1}{2}|\varrho_{ab} + \varrho_{bc}|^2)\,,\\ \Lambda_2 &= \tfrac{1}{2}\alpha_0(\varrho_{bb} + \varrho_{cc} - \tfrac{1}{2}|\varrho_{ab} + \varrho_{bc}|^2) + \tfrac{1}{2}\gamma\,,\\ \Lambda_3 &= \Lambda_4 = -\tfrac{1}{2}\alpha_0[\varrho_{ac} - \tfrac{1}{2}(\varrho_{ab} + \varrho_{bc})^2]\,. \end{aligned} \tag{2}$$

Here $\alpha_0 = 2 r_a g^2/\Gamma^2$ is the linear gain coefficient, γ the cavity loss rate, $s = r_a g/\Gamma$, r_a the atomic injection rate, g the atom-field coupling constant (for simplicity taken to be the same for the $a-b$ and $b-c$ transitions), Γ the atomic decay rate (same for all levels). $\varrho_{\alpha\beta}(\alpha\beta = a, b, c)$ are the initial atomic populations and coherence. Notice that f, Λ_3 and Λ_4 depend on the input coherences of the active atoms and the corresponding terms are phase sensitive, and lead to correlated emission and phase locking and thus give rise to quantum noise quenching and even squeezing. Here we consider the one photon resonance case ($\omega_{ab} = \omega_{bc} = \nu$).

A very useful way to study the properties of the field is to transform the master into a Fokker-Planck equation in some representation. Most widely used representations are Sudarshan-Glauber *P*, *Q* and Wigner representations. The Fokker-Planck equation in the three representations is the same:

$$\begin{aligned}\frac{\partial \Phi(\alpha, \alpha^*)}{\partial t} &= \left(-\frac{\partial}{\partial \alpha} d_\alpha - \frac{\partial}{\partial \alpha^*} d_\alpha^* + 2\frac{\partial^2}{\partial\alpha\,\partial\alpha^*} D_{\alpha^*\alpha}\right.\\ &\quad \left. + \frac{\partial^2}{\partial \alpha^2} D_{\alpha\alpha} + \frac{\partial^2}{\partial \alpha^{*2}} D_{\alpha\alpha}^*\right)\Phi(\alpha, \alpha^*)\,, \end{aligned} \tag{3}$$

where Φ stands for P, Q, or W.

From the relation between ϱ and P, Q, or W, we can find the roles of transformation from master equation to F–P equations, and consequently obtain the drift and diffusion coefficients. Two alternative forms of F–P equations are commonly used:

A) Polar Coordinates ($\alpha = r\,e^{i\phi}$),

$$\begin{aligned}\frac{\partial \Phi(r, \phi)}{\partial t} &= \left(-\frac{1}{r}\frac{\partial}{\partial r} r\,d_r - \frac{\partial}{\partial \phi} d_\phi + \frac{1}{r}\frac{\partial^2}{\partial r^2} r\,D_{rr}\right.\\ &\quad \left. + \frac{\partial^2}{\partial \phi^2} D_{\phi\phi} + \frac{2}{r}\frac{\partial^2}{\partial r\,\partial\phi} r\,D_{r\phi}\right)\Phi(r, \phi) \end{aligned} \tag{4}$$

with the drift coefficients (the same for P, Q and W)

$$\begin{aligned} d_r &= \mathrm{Re}(d_\alpha e^{-i\phi}) + r D_{\phi\phi}\,,\\ d_\phi &= \frac{1}{r}\,\mathrm{Im}(d_\alpha e^{-i\phi}) - 2 D_{r\phi}/r \end{aligned} \tag{5}$$

and diffusion coefficients listed in Table 1.

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Table 1. The diffusion coefficients in terms of amplitude and phase variables, r and ϕ, in the P, Q, and W representations.

Φ	P	Q	W
D_{rr}	$\frac{1}{2}[\mathrm{Re}\,A_1 + \lvert A_4\rvert \cos(\theta_4 - 2\phi)]$	$\frac{1}{2}[\mathrm{Re}\,A_2 + \lvert A_3\rvert \cos(\theta_3 - 2\phi)]$	$\frac{1}{2}(D^P_{rr} + D^Q_{rr})$
$D_{\phi\phi}$	$[\mathrm{Re}\,A_1 - \lvert A_4\rvert \cos(\theta_4 - 2\phi)]/2r^2$	$[\mathrm{Re}\,A_2 - \lvert A_3\rvert \cos(\theta_3 - 2\phi)]/2r^2$	$\frac{1}{2}(D^P_{\phi\phi} + D^Q_{\phi\phi})$
$D_{r\phi}$	$(2r)^{-1}\lvert A_4\rvert \sin(\theta_4 - 2\phi)$	$(2r)^{-1}\lvert A_3\rvert \sin(\theta_3 - 2\phi)$	$\frac{1}{2}(D^P_{r\phi} + D^Q_{r\phi})$

Table 2. The drift and diffusion coefficients in terms of the quadrature variables x and y (see (2.7)) in the P, Q, and W representations.

Φ	P	Q	W
d_x	$x\,\mathrm{Re}(A_1-A_2-A_3+A_4) - y\,\mathrm{Im}(A_1-A_2+A_3-A_4) + \mathrm{Re}\,f$		
d_y	$x\,\mathrm{Im}(A_1-A_2-A_3+A_4) + y\,\mathrm{Re}(A_1-A_2+A_3-A_4) + \mathrm{Im}\,f$		
D_{xx}	$\frac{1}{2}\mathrm{Re}(A_1+A_4)$	$\frac{1}{2}\mathrm{Re}(A_2+A_3)$	$\frac{1}{4}\mathrm{Re}(A_1+A_2+A_3+A_4)$
D_{yy}	$\frac{1}{2}\mathrm{Re}(A_1-A_4)$	$\frac{1}{2}\mathrm{Re}(A_2-A_3)$	$\frac{1}{4}\mathrm{Re}(A_1+A_2-A_3-A_4)$
D_{xy}	$\frac{1}{2}\mathrm{Im}\,A_4$	$\frac{1}{2}\mathrm{Im}\,A_3$	$\frac{1}{2}\mathrm{Im}(A_3+A_4)$

Table 3. The relations among the variances and covariance of the field quadratures with those of the Sudarshan-Glauber P, the Q, and the Wigner W distributions.

Φ	P	Q	W
$\langle(\Delta a_1)^2\rangle$	$\langle(\delta x)^2\rangle + \frac{1}{4}$	$\langle(\delta x)^2\rangle - \frac{1}{4}$	$\langle(\delta x)^2\rangle$
$\langle(\Delta a_2)^2\rangle$	$\langle(\delta y)^2\rangle + \frac{1}{4}$	$\langle(\delta y)^2\rangle - \frac{1}{4}$	$\langle(\delta y)^2\rangle$

B) Quadrature Coordinates
($\alpha = x + y$ and $a = a_1 + i a_2$),

$$\frac{\partial \Phi(x, y)}{\partial t} = \left(-\frac{\partial}{\partial x} d_x - \frac{\partial}{\partial y} d_y + \frac{\partial^2}{\partial x^2} D_{xx} + \frac{\partial^2}{\partial y^2} D_{yy} + \frac{\partial^2}{\partial x \partial y} 2 D_{xy} \right) \Phi(x, y) \tag{6}$$

with the drift and diffusion coefficient listed in Table 2.

Using the Fokker-Planck equation (6), the variances in the two quadratures x and y obey the following equations of motion:

$$\frac{d}{dt}\langle(\delta x)^2\rangle = 2A_{xx}\langle(\delta x)^2\rangle + 2A_{xy}\langle\delta x\,\delta y\rangle + 2D_{xx},$$

$$\frac{d}{dt}\langle(\delta y)^2\rangle = 2A_{yx}\langle\delta x\,\delta y\rangle + 2A_{yy}\langle(\delta y)^2\rangle + 2D_{yy},$$

$$\frac{d}{dt}\langle\delta x\,\delta y\rangle = (A_{xx} + A_{yy})\langle\delta x\,\delta y\rangle + A_{yx}\langle(\delta x)^2\rangle + A_{xy}\langle(\delta y)^2\rangle + 2D_{xy}, \tag{7}$$

where $A_{ij} = \partial d_i/\partial j$ $(i, j = x, y)$. The relation between $\langle(\delta x)^2\rangle$ and $\langle(\Delta a_1)^2\rangle$, and $\langle(\delta y)^2\rangle$ and $\langle(\Delta a_2)^2\rangle$ are listed in Table 3.

Choosing $\theta_{ab} = \theta_{bc} = \pi/2$ $(\theta_{ac} = \pi)$, we can find that all A's are real and $A_{xy} = A_{yx} = D_{xy} = 0$, which means that a_1 and a_2 $(a = a_1 + i a_2)$ are the amplitude and phase quadrature operators, respectively. Solving (6) at steady state, we obtain

$$\Phi = \frac{1}{2\pi\sqrt{\sigma_1\sigma_2}} \exp\left[-\frac{(x - x_0)}{2\sigma_1} - \frac{y^2}{2\sigma_2} \right] \tag{8}$$

with $x_0 = f/A_2 - A_1 + A_3 - A_4$, $\sigma_x = D_{xx}/(A_2 - A_1)$, and $\sigma_y = D_{yy}/(A_2 - A_1)$. This is a Gaussian distribution with widths σ_x and σ_y. It has been proven [1, 4] that the stable steady state condition is $A_2 - A_1 > 0$. Therefore, the sign of σ_x (or σ_y) is determined by D_{xx} (or D_{yy}). Using (2), we find the diffusion coefficients:

A) In Sudarshan-Glauber P

$$D^P_{xx} = \frac{1}{4}[\varrho_{aa} + \varrho_{bb} + \lvert\varrho_{ac}\rvert - (\lvert\varrho_{ab}\rvert + \lvert\varrho_{bc}\rvert)^2], \tag{9a}$$

$$D^P_{yy} = \frac{1}{4}(\varrho_{aa} + \varrho_{bb} - \lvert\varrho_{ac}\rvert). \tag{9b}$$

B) In Q

$$D^Q_{xx} = \frac{1}{4}\left[\varrho_{aa} + \varrho_{cc} + \lvert\varrho_{ac}\rvert + \frac{\gamma}{\alpha_0} - (\lvert\varrho_{ab}\rvert + \lvert\varrho_{bc}\rvert)^2\right], \tag{10a}$$

$$D^Q_{yy} = \frac{1}{4}\left(\varrho_{cc} + \varrho_{bb} + \frac{\gamma}{\alpha_0} - \lvert\varrho_{ab}\rvert\right). \tag{10b}$$

C) In Wigner

$$D^W_{yy} = D^P_{xx} + D^Q_{xx}, \quad D^W_{yy} = \frac{1}{4}\left(0.5 + \frac{1}{2}\varrho_{bb} - \lvert\varrho_{ac}\rvert\right). \tag{11}$$

It is clear that under the stable steady state condition all D's are positive, except D^P_{yy}. It is known that negative width means squeezing. Therefore, in Sudarshan-Glauber P representation we can know whether squeezing is possible or not by examining the signs of the diffusion coefficients, in the present case (A's real) by the signs of D_{xx} and D_{yy}. In the general case (Λ's complex) it is determined by the sign of $D_{\phi\phi}$ (or D_{rr}). This is a big advantage of the Sudarshan-Glauber P representation.

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[1] M. O. Scully, K. Wódkiewicz, M. S. Zubairy, J. Bergou, N. Lu, and J. Meyer ter Vehn, Phys. Rev. Lett. **60,** 1832 (1988).
[2] S. Y. Zhu and N. Lu, Phys. Lett. A **138,** 55 (1989).
[3] N. Lu, S. Y. Zhu, and G. S. Agarwal, Phys. Rev. A **40,** 258 (1989).
[4] N. Lu and S. Y. Zhu, Phys. Rev. A **40,** 5735 (1989).